King Fahd University of Petroleum and Minerals

College of Computer Science and Engineering
Information and Computer Science Department

ICS 253: Discrete Structures I
Summer Semester 2015-2016
Major Exam #2, Saturday August 20, 2016.

Name:

ID#:

Instructions:

1. This exam consists of **11** pages, including this page and an additional separate helping sheet, containing **five** questions.
2. You have to answer all **five** questions.
3. The exam is closed book and closed notes. No calculators or any helping aides are allowed. Make sure you turn off your mobile phone and keep it in your pocket if you have one.
4. The questions are **NOT equally weighed**. Some questions count for more points than others.
5. The maximum number of points for this exam is **100**.
6. You have exactly **100** minutes to finish the exam.
7. Make sure your answers are **readable**.
8. If there is no space on the front of the page, feel free to use the back of the page. Make sure you indicate this in order for me not to miss grading it.

Question Number	Maximum # of Points	Earned Points
1	**18**	
2	**17**	
3	**30**	
4	**20**	
5	**15**	
Total	**100**	

I. (18 points)

1. (8 points) Prove that if $x, y \in \Re$ then $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$ or 1.

Let $x = m + \epsilon_1$, $y = n + \epsilon_2$ where $m, n \in \mathbb{Z}$ & $\epsilon_1, \epsilon_2 \in \mathbb{R}$
$0 < \epsilon_1 \leq 1$, $0 < \epsilon_2 \leq 1$.

Then, $\lceil x \rceil = \lceil m + \epsilon_1 \rceil = m + 1$.

Similarly, $\lceil y \rceil = n + 1$. $\therefore \lceil x \rceil + \lceil y \rceil = m + n + 2$

Now, $\lceil x + y \rceil = \lceil (m + \epsilon_1) + (n + \epsilon_2) \rceil = \lceil (m+n) + (\epsilon_1 + \epsilon_2) \rceil$.

Since $0 < \epsilon_1 \leq 1$ & $0 < \epsilon_2 \leq 1$, $0 < \epsilon_1 + \epsilon_2 \leq 2$.

$\therefore \lceil x + y \rceil = m + n + 1$ or $m + n + 2$

$$\therefore \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = \begin{cases} m+n+2 - (m+n+1) = 1 \\ m+n+2 - (m+n+2) = 0 \end{cases}$$

2. (10 points) Compute the following summation:

$$S = \sum_{i=100}^{1000} \sum_{j=1}^{50} (i.2^j + j^2)$$

$$S = \sum_{i=100}^{1000} \sum_{j=1}^{50} i.2^j + \sum_{i=100}^{1000} \sum_{j=1}^{50} j^2$$

$$= \sum_{i=100}^{1000} i \sum_{j=1}^{50} 2^j + \sum_{i=100}^{1000} \left(\frac{(50)(51)(101)}{6} \right)$$

$$= \sum_{i=100}^{1000} i \left[\sum_{j=0}^{50} 2^j - \sum_{j=0}^{0} 2^j \right] + \frac{(1000-100+1)(50)(51)(101)}{6}$$

$$= \sum_{i=100}^{1000} i \left[\frac{2^{50+1} - 1}{2-1} - 1 \right] + \frac{(50)(51)(101)(901)}{6}$$

$$= (2^{51} - 2) \sum_{i=100}^{1000} i + \frac{(50)(51)(101)(901)}{6}$$

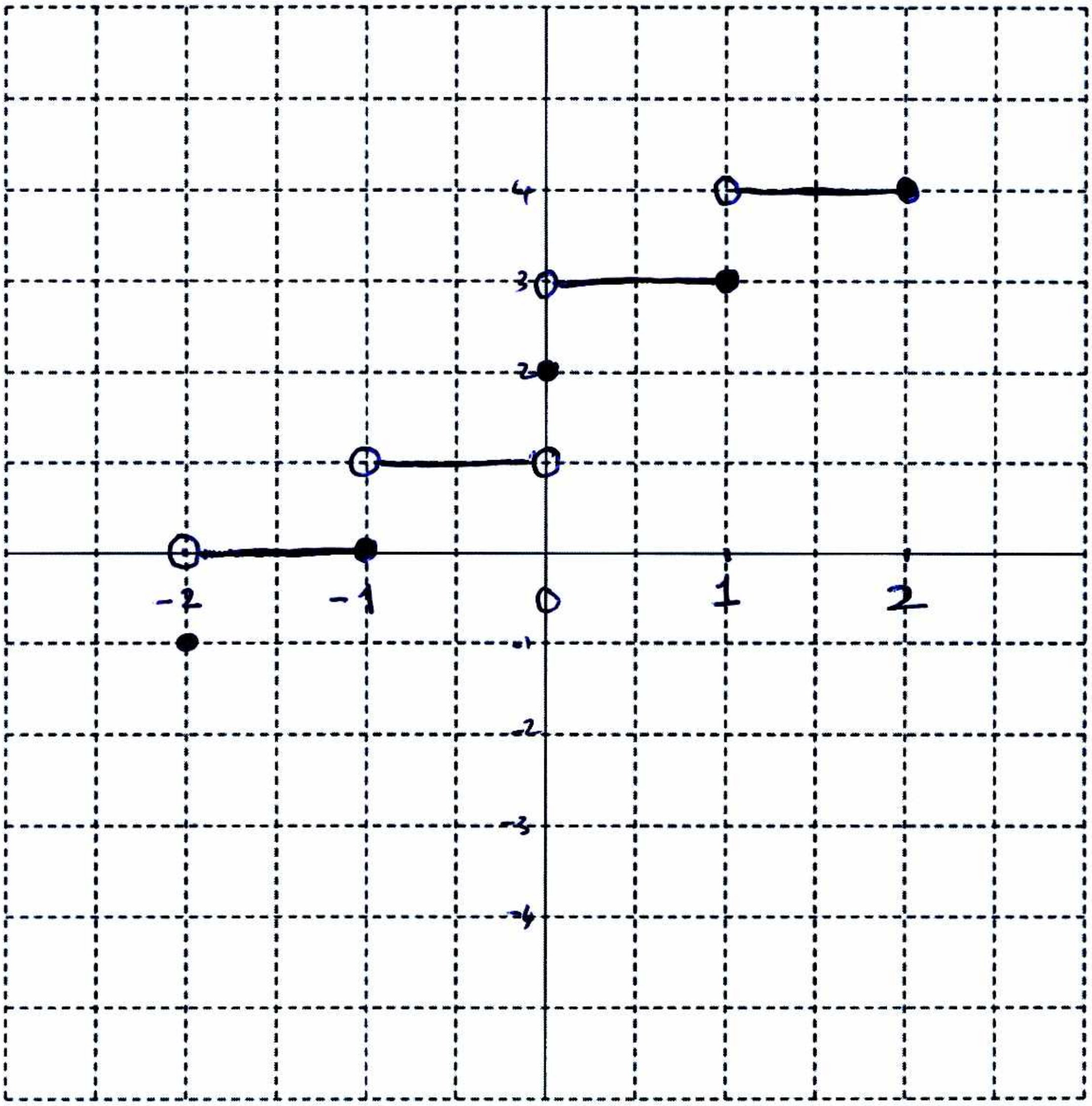
$$= (2^{51} - 2) \left(\sum_{i=1}^{1000} i - \sum_{i=1}^{99} i \right) + \frac{(50)(51)(101)(901)}{6}$$

$$= (2^{51} - 2) \left(\frac{(1000)(1001)}{2} - \frac{99(100)}{2} \right) + \frac{(50)(51)(101)(901)}{6}$$

II. **(17 points)**

1. (10 points) Draw the graph of the function

$$f(x) = \left\lfloor \frac{x}{3} \right\rfloor + \lceil x + 2 \rceil \qquad \text{where } -2 \le x \le 2$$



$x \in$	$\lfloor \frac{x}{3} \rfloor$	$\lceil x+2 \rceil$	$f(x)$
$\{-2\}$	-1	0	-1
$(-2,-1]$	-1	1	0
$(-1,0)$	-1	2	1
$\{0\}$	0	2	2
$(0,1]$	0	3	3
$(1,2]$	0	4	4

2. (7 points) Prove that the set of real numbers with decimal representations of all 2's, 4's or 5's is uncountable.

Assume they are countable. Hence, we can list them as and consider those in $[0,1]$

$r_1 = 0.d_{11}\, d_{12}\, d_{13} \cdots\cdots$

$r_2 = 0.d_{21}\, d_{22}\, d_{23} \cdots\cdots$

$r_3 = 0.d_{21}\, d_{32}\, d_{33} \cdots\cdots$

$\vdots$

$r_i = 0.d_{i1}\, d_{i2}\, d_{i3} \cdots\cdots$

$\vdots$

Now construct the following real number x consisting of only numbers that are decimals of the form as follows

$x = 0.d_1\, d_2\, d_3 \cdots$

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

Obviously $x \notin \{r_1, r_2, \ldots, r_i, \ldots\}$

Since for any real number r_j in the above set, r_j differs with x in the j th decimal. (if $d_{jj} = 4$ then $d_j = 5$, if $d_{jj} \in \{2,5\}$ then $d_j = 4$.

$\ast$ contradiction $\ast$.

Since the set of real numbers $\in [0,1]$ with 2's, 4's & 5's decimal representation containing is uncountable, then real numbers with decimal representation of 2s, 4s & 5s is uncountable.

III. **(30 points)**

1. (5 points) Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if $a_n = 4n - 2$.

$$a_{n+1} = 4(n+1) - 2 = 4n + 4 - 2$$
$$= (4n - 2) + 4$$
$$= a_n + 4$$

$$\therefore \quad a_{n+1} = \begin{cases} 2 & n = 0 \\ a_n + 4 & n > 0 \end{cases}$$

2. (5 points) Give a recursive definition of the set of positive integer powers of 3.

Let the Set be called S'. Then

$3 \in S'$

if $x, y \in S'$ then $x * y \in S'$.

3. (10 points) Prove that $H_{2^n} \leq 1 + n$ whenever n is a nonnegative integer, and $H_i = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}$ is the harmonic series.

We use mathematical induction.

Basis: $n = 0$.

$$H_{2^0} = H_1 = \underbrace{1 \overset{?}{\leq} 1 + 0}$$ holds ✓.

Inductive step: Assume that $H_{2^n} \leq 1 + n$. to show that $H_{2^{n+1}} \leq 1 + (n+1) = n+2$.

$$H_{2^{n+1}} = H_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \quad (2^n + 2^n = 2^{n+1})$$

$$\leq (1+n) + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} \quad \text{(By Induction Hypothesis)}$$

$$\leq 1 + n + \underbrace{\left(\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)}_{2^n \text{ terms}}$$

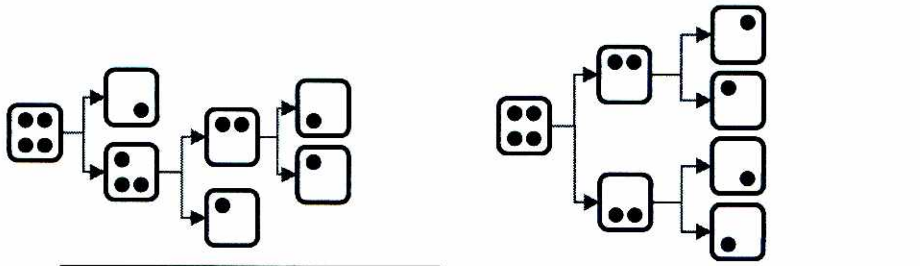
$$\left(\text{Since } \frac{1}{2^n+i} < \frac{1}{2^n} \; \forall i \in \{1, 2, 3, \ldots\}\right).$$

$$= 1 + n + 2^n \left(\frac{1}{2^n}\right)$$

$$= 1 + n + 1$$

$$= n + 2.$$

4. (10 points) Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n-1)/2$ using strong induction. The Figure below explains the question by showing all splits of 4 stones.



(1)(3) + (2)(1) +(1)(1) = 6

(2)(2) +(1)(1) +(1)(1) = 6

You can also consider the case when n=1.

Basis: $n = 2$.

The only way to split them is $1+1$.

hence $(1)(1) = 1 \stackrel{?}{=} \frac{2(1)}{2} = 1$ ✓ holds

Induction step: Assume that the result holds for all k $2 \le k \le n$. To prove it for $n+1$, i.e. $P(n+1) = \frac{(n+1)n}{2}$.

Let us split $n+1$ into two piles, j & $(n+1)-j$ where $1 \le j \le n$

Applying the induction hypothesis if $j > 1$ and $j < n$, we get

$$j \rightarrow \frac{j(j-1)}{2} \qquad (n+1-j) \longrightarrow \frac{(n+1-j)(n-j)}{2}$$

$$\therefore P(n+1) = \frac{j(j-1)}{2} + \frac{(n+1-j)(n-j)}{2} + j(n+1-j)$$

$$= \frac{j^2 - j + n^2 - nj + n - j - jn + j^2 + 2j(n+1-j)}{2}$$

$$= \frac{\left[j^2 - j + n^2 - 2nj + n - j + j^2\right] + \left[2nj + 2j - 2j^2\right]}{2}$$

$$= \frac{\left[2j^2 - 2j - 2nj + n + n^2\right] + \left[2nj + 2j - 2j^2\right]}{2}$$

$$= \frac{n + n^2}{2} = \frac{n(n+1)}{2} = \frac{(n+1)n}{2}.$$

Note that if $j = 1$ or $j = n$,
Then, the piles are split as n & 1.
Applying induction hypothesis will be only on n.

$$\text{Hence } P(n+1) = \frac{n(n-1)}{2} + (n)(1)$$

$$= \frac{n(n-1)}{2} + n$$

$$= \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{(n+1)n}{2}.$$

IV. (20 points)

1. (3 points) How many strings of three decimal digits begin with an odd digit?

$(5)(10)(10)$

2. (4 points) How many subsets of a set with 50 elements have more than one element?

The size of the set of all subsets <u>minus</u>
(the empty set & sets with 1 elt)

$= 2^{50} - 1 - 50 = 2^{50} - 51.$

3. (8 points) How many bit strings of length 12 contain
   a. an equal number of 0s and 1s?

$\binom{12}{6}$

   b. at least three 1s?

Same as the number of all bit strings of length 12
− (bit strings having 0 1s, 1 1 and 2 1s).

$= 2^{12} - \left(\binom{12}{0} + \binom{12}{1} + \binom{12}{2}\right)$

4. (5 points) A club has 25 members. How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

$P(25,4) = (25)(24)(23)(22).$

V. (15 points)

1. (4 points) Prove that if six integers are selected from the first 10 positive integers, there must be at least one pair of integers with the sum 11.

Consider the following pairs :

$(1,10), (2,9), (3,8), (4,7), (5,6)$. all adding up to 11.

We divided the 10 +ve integers into 5 pairs!

By the Pigeon hole principle, choosing 6 numbers mean that at least 2 are chosen from the same pair & hence they add up to 11.

2. (6 points) Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Prove your answer.

C_1 —— P_1

C_2 —— P_2

C_3 —— P_3

C_4 —— P_4

$C5$ (all)

$C6$ (all)

$C7$ (all)

$C8$ (all)

Least # of cables

$= 4 + 4(4) = 20.$

Note that any 4 computers using these 20 cables can connect simultaneously to 4 printers. Assume we can go by 19 cables. By the generalized pigeon hole principle, at least one printer is connected to 4 computers (if not (i.e. all are 5, then we have $5(4) = 20$ cables *contradiction*)

Hence the rest 4 computers have access to only 3 computers * contradiction *

3. (5 points) Compute the value of

$$\sum_{i=1}^{1000} \binom{1000}{i} 3^i (-2)^{1000-i}$$

Since $$\sum_{i=0}^{1000} \binom{1000}{i} 3^i (-2)^{1000-i} = (3-2)^{1000}$$

$$= 1^{1000}$$

$$= 1.$$

$$\therefore \sum_{i=1}^{1000} \binom{1000}{i} 3^i (-2)^{1000-i} = 1 - \binom{1000}{0} 3^0 (-2)^{1000}$$

$$= -2^{1000} + 1$$